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ON THE POLES OF IGUSA'S LOCAL ZETA FUNCTION FOR ALGEBRAIC SETS

W. A. ZUNIGA-GALINDO

Abstract. Let K be a $p-$adic field, and $Z_{\Phi}(s,f)$, $s\in\mathbb{C}$, with $Re(s)>0$, the Igusa local zeta function associated to $f(x)=(f_1(x),..,f_l(x))\in[K(x_1,..,x_n)]^l$, and Φ a Schwartz-Bruhat function. The aim of this paper is to describe explicitly the poles of the meromorphic continuation of $Z_{\Phi}(s,f)$. Using resolution of singularities is possible to express $Z_{\Phi}(s,f)$ as a finite sum of $p-$adic monomial integrals. These monomial integrals are computed explicitly by using techniques of toroidal geometry. In this way, an explicit list for the candidates to poles of $Z_{\Phi}(s,f)$ is obtained.

1. Introduction

Let K be a $p-$adic field, i.e. $[K:\mathbb{Q}_p]<\infty$. Let $\mathcal{O}_K$ be the valuation ring of K, $\mathcal{P}_K$ the maximal ideal of $\mathcal{O}_K$, and $\overline{K}=\mathcal{O}_K/\mathcal{P}_K$ the residue field of K. The cardinality of the residue field of K is denoted by q, thus $\overline{K}=\mathbb{F}_q$. For $z\in K$, $v(z)\in\mathbb{Z}\cup\{+\infty\}$ denotes the valuation of z, $|z|_K=q^{-v(z)}$, and $ac\ z=z\pi^{-v(z)}$ where π is a fixed uniformizing parameter for $\mathcal{O}_K$. For $x=(x_1,...,x_l)\in K^l$, we define $\|x\|_K:=\max_{1\leq i\leq l}|x_i|_K$.

Let $f_i(x)\in K[x]$, $x=(x_1,...,x_n)$, be a non-constant polynomial for $i=1,...,l$. Let $\Phi: K^n\rightarrow\mathbb{C}$ be a Schwartz-Bruhat function, i.e. a locally constant function with compact support. The Igusa local zeta function attached to $f(x):=(f_1(x),...,f_l(x))$ is defined as

$$Z_{\Phi}(s,f)=\int_{K^n}\Phi(x)\,\|f(x)\|_K^s\mid dx\mid,\qquad s\in\mathbb{C},\tag{1.1}$$

for $Re(s)>0$, where $\mid dx\mid$ denotes the Haar measure on K^n so normalized that $\mathcal{O}_K^n$ has measure 1. Since the numerical data of a resolution of singularities of $\cup_{i=1}^l f_i^{-1}(0)$ are directly related to the meromorphic continuation of $Z_{\Phi}(s,f)$, we say that $Z_{\Phi}(s,f)$ is the local zeta function associated to $\cup_{i=1}^l f_i^{-1}(0)$. Alternatively, we may say that $Z_{\Phi}(s,f)$ is the local zeta function associated to the polynomial mapping $f:K^l\rightarrow K^n$.

The functions $Z_{\Phi}(s,f)$ were introduced by Weil [18] and their basic properties for general f and $l=1$ were first studied by Igusa. Using resolution of singularities Igusa proved that $Z_{\Phi}(s,f)$ admits a meromorphic continuation to the complex plane as a rational function of q^{-s} [8], [10, Theorem 8.2.1]. Igusa's proof was generalized by Meuser to the case $l\geqslant 1$ [14, Theorem 1]. Using $p-$adic cell decomposition,

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Denef gave a completely different proof of the rationality of $Z_\Phi(s,f)$, with $l \geqslant 1$ [2].

We write $Z(s,f)$ when Φ is the characteristic function of $\mathcal{O}_K^n$. Suppose that $f_i(x)$, $i=1,...,l$, have coefficients in $\mathcal{O}_K$. Let $N_j(f)$ be the number of solutions of $f_i(x) \equiv 0 \mod \mathcal{P}_K^j$, $i=1,...,l$, in $\left(\mathcal{O}_K/\mathcal{P}_K^j\right)^n$, and put $P(t,f) := \sum_{j=0}^{\infty} N_j(f)(q^{-n}t)^j$. The Poincaré series $P(t,f)$ is related to $Z(s,f)$ by the formula

$$P(t,f) = \frac{1-tZ(s,f)}{1-t}, \quad t = q^{-s}, \tag{1.2}$$

[14, Theorem 2]. Thus the rationality of $Z(s,f)$ implies the rationality of $P(t,f)$. In the case $l=1$ the rationality of $P(t,f)$ was conjectured by S. E. Borewicz and I. R. Šafarevič [1, page 63]. The Borewicz-Šafarevič conjecture is a consequence of the rationality of $Z(s,f)$ proved by Igusa, Meuser, and Denef.

A basic problem is to determine the poles of the meromorphic continuation of $Z_\Phi(s,f)$ in $\mathrm{Re}(s) < 0$. In the case $l=1$, the basic strategy is to take a resolution of singularities $h: X_K \to A_K^n$ of $f^{-1}(0)$ and study the numerical data $\{(N_i, v_i)\}$ of the resolution, where N_i is the multiplicity of f along the exceptional divisor D_i, and $\nu_i - 1$ is the multiplicity of $h^*(\bigwedge_{1\leq j\leq n} dx_j)$ along the divisor D_i. Igusa showed that the set of ratios $\left\{-\frac{v_i}{N_i}\right\} \cup \{-1\}$ contains the real parts of the poles of $Z_\Phi(s,f)$ [8], [10, Theorem 8.2.1]. Many examples show that almost all these fractions do not correspond to poles of $Z_\Phi(s,f)$. In the case of polynomials in two variables, as a consequence of the works of Igusa, Strauss, Meuser and Veys, there is a complete solution of this problem [9], [15], [13], [16], [17]. For general polynomials the problem of determination of the poles of $Z_\Phi(s,f)$ is still open. There exists a generic class of polynomials called non-degenerate with respect to its Newton polyhedron for which is possible to give a small set of candidates for the poles of $Z_\Phi(s,f)$. The poles of the local zeta functions attached to non-degenerate polynomials can be described in terms of Newton polyhedra. The case of two variables was studied by Lichtin and Meuser [11]. In [3], Denef gave a procedure based on monomial changes of variables to determine a small set of candidates for the poles of $Z_\Phi(s,f)$ in terms of the Newton polyhedron of f. This result was obtained by the author, using an approach based on the $p-$adic stationary phase formula and Néron $p-$desingularization, for polynomials with coefficients in a non-archimedean local field of arbitrary characteristic [20], (see also [4], [19], [21]).

The local zeta functions $Z_\Phi(s,f)$, with $l \geqslant 1$, are a generalization of the Igusa local zeta functions associated to hypersurfaces. Another generalizations are due to Denef and Loeser. In [12] Loeser introduced a local zeta function in l complex variables associated to an algebraic set of the form D_K. Recently, Denef and Loeser have introduced a completely new class of zeta functions called motivic Igusa zeta functions [5] that include as particular cases the Igusa local zeta functions, with $l=1$, and the topological zeta functions [6].

The first result of this paper (see theorem 4.1) provides an explicit description of the poles of $Z_\Phi(s,f)$, with $l \geqslant 1$, in terms of a resolution of singularities of f.

Using resolution of singularities it is possible to express $Z_\Phi(s,f)$ as a finite sum of $p-$adic monomial integrals. We compute explicitly these monomial integrals by using toroidal geometry (see section 3). In this way we are able to give an explicit list for the candidates to poles of $Z_\Phi(s,f)$ in terms of a list of Newton polyhedra constructed from the numerical data associated to a resolution of singularities of the divisor $\cup_{i=1}^{l} f_i^{-1}(0)$. The second result provides an explicit description of the poles of a twisted local zeta function $Z_\Phi(s,\chi,f)$ in terms of a resolution of singularities of f (see theorem 5.1). Finally, we give a family of algebraic sets, that we called monomial algebraic sets, for which the corresponding local zeta functions can be expressed in terms of Newton polyhedra (see theorem 6.1).

2. Preliminaries

2.1. **Numerical data.** We put $D_K := \cup_{i=1}^{l} f_i^{-1}(0)$. By Hironaka's resolution theorem [7] applied to D_K, there exist an $n-$dimensional $K-$analytic manifold X_K and proper $K-$analytic map $h : X_K \to A_K^n$ with the following properties:

(1) $h^{-1}(D_K)$ is a divisor with normal crossings;

(2) the restriction of h to $X_K \setminus h^{-1}(D_K)$ is an isomorphism onto its image.

We denote by E_i, $i \in T$, the irreducible components of the divisor $h^{-1}(D_K)$. At every point b of X_K if E_1, E_2,..., E_r, $r \leq n$, are all the components containing b with respective local equations y_1,...,y_r around b, then there exist local coordinates of X_K around b of the form $(y_1,...,y_r,y_{r+1},...,y_n)$ such that

$$f_i \circ h = \epsilon_i \prod_{1\leq j\leq r} y_j^{N_{i,j}},\ i = 1,...,l, \tag{2.1}$$

$$h^*\Big(\bigwedge_{1\leq j\leq n} dx_j\Big) = \left(\eta \prod_{1\leq j\leq r} y_j^{v_j-1}\right) \bigwedge_{1\leq j\leq n} dy_j, \tag{2.2}$$

on some neighborhood of b, in which ϵ_i, $i = 1,...,l$, η are units of the local ring $\mathcal{O}_b$ of X_K at b.

We set

$$\begin{aligned} N_i &:= (N_{i,1},...,N_{i,r},0,..,0) \in \mathbb{N}^n, \\ N_b &:= \{N_1,...,N_l\}, \end{aligned} \tag{2.3}$$

and

$$v_b := (v_1-1,...,v_r-1,0,...,0) \in \mathbb{N}^n. \tag{2.4}$$

We define the *numerical data* of $f = (f_1,...,f_l)$ as the set of all pairs (N_b, v_b) satisfying $N_i \neq 0$, $i = 1,...,l$, and $b \in X_K$.

We note that if (N_b', v_b') is a numerical datum at $b \in X_K$ obtained by using a second local coordinate system, then it may be verified that $N_b = N_b'$, $v_b = v_b'$, after a permutation of indices, i.e. after a renaming of the local coordinates around b. As we shall see later on our description of the poles of $Z_\Phi(s,f)$ does not depend on the local coordinate systems used in the computation of the numerical data of f.

2.2. **Newton polyhedra.** Let $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geqslant 0\}$, and S a non-empty subset of $\mathbb{N}^n$. The *Newton polyhedron* $\Gamma(S)$ associated to S is defined as the convex hull in $\mathbb{R}_+^n$ of the set $\bigcup_{m\in S}\left(m+\mathbb{R}_+^n\right)$. A *facet* is a face of $\Gamma(S)$ of dimension $n-1$.

We denote by $\langle\ .\ ,.\ \rangle$ the usual inner product of $\mathbb{R}^n$. For $a \in \mathbb{R}_+^n$, we define

$$m(a) := \inf_{x\in\Gamma(S)} \langle a, x\rangle\,. \tag{2.5}$$

Given $a \in \mathbb{R}_+^n$, the *first meet locus* $F(a)$ of a is defined as

$$F(a) := \{x \in \Gamma(S) \mid \langle a, x\rangle = m(a)\}. \tag{2.6}$$

The first meet locus is a face of $\Gamma(S)$. Moreover, if $a \neq 0$, $F(a)$ is a proper face of $\Gamma(S)$.

We define an equivalence relation in $\mathbb{R}_+^n$ by

$$a \sim a' \quad \text{iff} \quad F(a) = F(a'). \tag{2.7}$$

The equivalence classes of $\sim$ are sets of the form

$$\Delta_\tau = \{a \in \mathbb{R}_+^n \mid F(a) = \tau\}, \tag{2.8}$$

where τ is a face of $\Gamma(S)$. We note that $\Delta_{\Gamma(S)} = \{0\}$.

We recall that the cone strictly spanned by the vectors $a_1, .., a_e \in \mathbb{R}_+^n \setminus \{0\}$ is the set $\Delta = \{\lambda_1 a_1 + ... + \lambda_e a_e \mid \lambda_i \in \mathbb{R}_+, \lambda_i > 0\}$. If $a_1, .., a_e$ are linearly independent over $\mathbb{R}$, Δ is called a *simplicial cone*. If moreover $a_1, .., a_e \in \mathbb{Z}^n$, we say Δ is a *rational simplicial cone*. If $\{a_1, .., a_e\}$ is a subset of a basis of the $\mathbb{Z}$-modulo $\mathbb{Z}^n$, we call Δ a *simple cone*.

A precise description of the geometry of the equivalence classes modulo $\sim$ is as follows. Each facet γ of $\Gamma(S)$ has a unique primitive vector $a(\gamma) \in \mathbb{R}^n$ (i.e. a vector whose components are positive integers and their greatest common divisor is one) which is perpendicular to γ. We denote by $\mathfrak{D}(\Gamma(S))$ the set of such vectors. The equivalence classes are exactly the cones

$$\Delta_\tau = \{\sum_{i=1}^{e} \lambda_i a(\gamma_i) \mid \lambda_i \in \mathbb{R},\, \lambda_i > 0\}, \tag{2.9}$$

where τ runs through the set of faces of $\Gamma(S)$, and γ_i, $i = 1, .., e$ are the facets containing τ. We note that $\Delta_\tau = \{0\}$ if and only if $\tau = \Gamma(S)$.

Each cone Δ_τ can be partitioned into a finite number of simple cones $\Delta_{\tau,i}$. In addition, the subdivision can be chosen such that each $\Delta_{\tau,i}$ is spanned by part of $\mathfrak{D}(\Gamma(S))$ (see e.g. [4, sect. 2]). Thus from the above considerations we have the following partition of $\mathbb{R}_+^n$:

$$\mathbb{R}_+^n = \{0\} \bigcup \bigcup_{\tau} \left(\bigcup_{i=1}^{l_\tau} \Delta_{\tau,i} \right), \tag{2.10}$$

where τ runs over the proper faces of $\Gamma(S)$, and each $\Delta_{\tau,i}$ is a simple cone contained in Δ_τ. We shall say that $\{\Delta_{\tau,i}\}$ is a *simple polyhedral subdivision of* $\mathbb{R}_+^n$ *subordinated* to $\Gamma(S)$.

3. $p-$ADIC MONOMIAL INTEGRALS AND NEWTON POLYHEDRA

We set $N_i = (N_{i,1}, ..., N_{i,n}) \in \mathbb{N}^n, N_i \neq 0$, for $i = 1, 2, ..., l$, $N = (N_1, ..., N_l)$, and $v = (v_1, ..., v_n) \in \mathbb{N}^n$. Let x^{M_i} denote the monomial $\prod_{j=1}^{n} x_j^{M_{i,j}}$, where $M_i = (M_{i,1}, ..., M_{i,n}) \in \mathbb{N}^n$.

We associate to the pair (N, v) the $p-$adic integral

$$I_{(N,v)}(s) := \int_{(\mathcal{P}_K^{e_0})^n} \left\| \left(c_1 x^{N_1}, ..., c_l x^{N_l}\right) \right\|_K^s |x^v|_K \, |dx| \,, \quad Re(s) > 0, \tag{3.1}$$

where $c_i \in \mathcal{O}_K$, $i = 1, 2, ..., l$, are constants, $e_0 \in \mathbb{N}$, and $|dx|$ is a Haar measure on K^n so normalized that $\mathcal{O}_K^n$ has measure one. In this section we show that $I_{(N,v)}(s)$ is a rational function of q^{-s} and describe its poles in terms of a Newton polyhedron.

We associate to (N, v) the Newton polyhedron $\Gamma(N)$. The set of primitive perpendicular vectors to the faces of $\Gamma(N)$ is denoted by $\mathfrak{D}(\Gamma(N))$. Each facet $\gamma \in \Gamma(N)$ is the intersection of $\Gamma(N)$ and a *supporting hyperplane* with equation $\langle a(\gamma), x\rangle = m(a(\gamma))$. If $a(\gamma) = (a_1, ..., a_n)$, we set $\sigma(a(\gamma)) := \sum_{i=1}^{n} a_i$.

Lemma 3.1. *The integral $I_{(N,v)}(s)$ is a rational function of q^{-s} with poles of the form*

$$s = -\frac{\sigma(a(\gamma)) + \langle v, a(\gamma)\rangle}{m(a(\gamma))} + \frac{2\pi\sqrt{-1}k}{m(a(\gamma))\log q}, \; k \in \mathbb{Z}, \tag{3.2}$$

where γ is a facet of $\Gamma(N)$, and $\langle a(\gamma), x\rangle = m(a(\gamma))$, with $m(a(\gamma)) \neq 0$, is the equation of the supporting hyperplane of γ.

Proof. First we fix some notation. Given $x = (x_1, ..., x_n) \in \mathcal{O}_K^n$, we set $v(x) := (v(x_1), ..., v(x_n))$, and

$$E_A := \{x \in (\mathcal{P}_K^{e_0})^n \mid v(x) \in A \cap \mathbb{N}^n\}, \tag{3.3}$$

for any $A \subseteq \mathbb{R}_+^n$.

We fix a simple polyhedral subdivision of $\mathbb{R}_+^n$ subordinated to $\Gamma(N)$. From this subdivision we get the following partition for $(\mathcal{P}_K^{e_0})^n$:

$$(\mathcal{P}_K^{e_0})^n = \bigcup_{\tau \subset \Gamma(N)} \bigcup_{i=1}^{l_\tau} E_{\Delta_{\tau,i}}, \text{ if } \quad e_0 \geqslant 1, \tag{3.4}$$

where τ is a proper face of $\Gamma(N)$, and $\Delta_{\tau,i}$ is a simple cone. In the case in which $e_0 = 0$, i.e. $(\mathcal{P}_K^{e_0})^n = \mathcal{O}_K^n$, the following holds

$$\mathcal{O}_K^n = \left(\mathcal{O}_K^\times\right)^n \bigcup \bigcup_{\tau \subset \Gamma(N)} \bigcup_{i=1}^{l_\tau} E_{\Delta_{\tau,i}}. \tag{3.5}$$

From the above partitions, it follows that $I_{(N,v)}(s)$ is a finite sum of integrals of the form

$$\int_{(\mathcal{O}_K^\times)^n} \left\| \left(c_1 x^{N_1}, ..., c_l x^{N_l}\right) \right\|_K^s |dx| = \|(c_1, ..., c_l)\|_K^s \,(1 - q^{-1})^n, \tag{3.6}$$

and

$$\int\limits_{E_{\Delta_{\tau,i}}} \left\| \left(c_1 x^{N_1}, ..., c_l x^{N_l}\right) \right\|_K^s |x^v|_K \, |dx| . \tag{3.7}$$

Without loss of generality, we assume that $e_0 \geqslant 1$. By virtue of the above considerations it is sufficient to show that integral (3.7) is a rational function of q^{-s} and that its poles have form (3.2).

On the other hand, since $\Delta_{\tau,i}$ is a simple cone, it is spanned by some $a_1, ..., a_e \in \mathfrak{D}(\Gamma(N))$, $e = e_{\tau,i}$, and $\{a_1, ..., a_e\}$ is a subset of a basis of the $\mathbb{Z}-$module $\mathbb{Z}^n$. Thus

$$\Delta_{\tau,i} \cap \mathbb{N}^n = \{\sum_{i=1}^{e} a_i y_i \mid y_i \in \mathbb{N} \setminus \{0\}\}. \tag{3.8}$$

We set $I = \{1, 2, ..., e\}$, $J \subseteq I$, and for a fixed $\alpha_0 \in \mathbb{N} \setminus \{0\}$,

$$A_J := \{\sum_{i=1}^{e} a_i y_i \mid y_i \in \mathbb{N} \setminus \{0\}, \text{ and } y_i \geqslant \alpha_0 + 1 \Leftrightarrow i \in J\},$$

$$E_{A_J} = \{x \in (\mathcal{P}_K^{e_0})^n \mid v(x) \in A_J\}.$$

We subdivide $\Delta_{\tau,i} \cap \mathbb{N}^n$ as

$$\Delta_{\tau,i} \cap \mathbb{N}^n = \bigcup_{J \subseteq I} A_J. \tag{3.9}$$

From partition (3.9), it follows that

$$\int\limits_{E_{\Delta_{\tau,i}}} = \sum_{J \subseteq I} \int\limits_{E_{A_J}} := \sum_{J \subseteq I} I(s, E_{A_J}). \tag{3.10}$$

The integral $I(s, E_{A_\emptyset})$ has an analytic continuation to the complex plane as a polynomial in q^{-s}. Hence it is sufficient to show that the integral $I(s, E_{A_J})$ is a rational function of q^{-s}, and that its poles have form (3.2), for $J \neq \emptyset$.

The integral $I(s, E_{A_J})$ admits the following series expansion:

$$I(s, E_{A_J}) = \int\limits_{E_{A_J}} \left\| \left(c_1 x^{N_1}, ..., c_l x^{N_l}\right) \right\|_K^s |x^v|_K \, |dx|$$

$$= \sum_{k \in A_J} \int\limits_{\{x \in \left(\mathcal{P}_K^{e_0}\right)^n \mid v(x) = k\}} \left\| \left(c_1 x^{N_1}, ..., c_l x^{N_l}\right) \right\|_K^s |x^v|_K \, |dx| . \tag{3.11}$$

We write $k = (k_1, ..., k_n) \in \mathbb{N}^n$, $\sigma(k) := k_1 + ... + k_n$, and

$$x_j = \pi^{k_j} u_j, \quad u_j \in \mathcal{O}_K^{\times}, \ j = 1, 2, ..., n. \tag{3.12}$$

Then

$$|dx| = q^{-\sigma(k)} |du|, \tag{3.13}$$
$$x^{N_j} = \pi^{\langle k, N_j \rangle} u^{N_j}.$$

We know that $F(k) = \tau$. Thus

$$\langle k, N_j \rangle = m(k), \quad \text{for every } N_j \in \{N_1, ..., N_l\} \cap \tau, \tag{3.14}$$

and

$$\langle k, N_j\rangle > m(k), \text{ for every } N_j \in \{N_1, ..., N_l\} \setminus \tau. \tag{3.15}$$

We may assume, possibly after a renaming of the N_j, that $\{N_1, ..., N_l\} \cap \tau = \{N_1, ..., N_r\}$, with $r \leq l$. Thus $\{N_1, ..., N_l\} \setminus \tau = \{N_{r+1}, ..., N_l\}$. With this notation, and since $k = \sum_{i=1}^{e} a_i y_i$, it follows from (3.13), (3.14), (3.15), that

$$x^{N_j} = \begin{cases} \pi^{\sum_{i=1}^{e} y_i m(a_i)} u^{N_j}, & j = 1, 2, ..., r, \\ \pi^{\sum_{i=1}^{e} y_i \langle a_i, N_j\rangle} u^{N_j}, & j = r+1, r+2, ..., l. \end{cases} \tag{3.16}$$

In addition, for $j = r+1, 2, ..., l$, it holds that $\langle a_i, N_j\rangle > m(a_i)$, for some index i.

We fix α_0 satisfying the condition:

$$\alpha_0 = e_0 + \max_{1\leq i\leq n} \{v(c_i)\}.$$

The constant α_0 was selected to guarantee that the following two conditions are satisfied:

(3.17)

$$E_{A_J} = \left\{ x \in (\mathcal{P}_K^{e_0})^n \mid v(x) = \sum_{i=1}^{e} a_i y_i, \text{ for some } y_i \in \mathbb{N}\setminus\{0\}, \, y_i \geqslant \alpha_0 + 1 \Leftrightarrow i \in J \right\}$$

for every $J \neq \emptyset$, and

$$\left\| \left(c_1 x^{N_1}, ..., c_l x^{N_l}\right) \right\|_K |_{E_{A_J}} = \left\| \left(c_1 x^{N_1}, ..., c_r x^{N_r}\right) \right\|_K |_{E_{A_J}} . \tag{3.18}$$

Then from (3.13), (3.16), and (3.18) it follows that

$$\begin{aligned} &\left\| \left(c_1 x^{N_1}, ..., c_l x^{N_l}\right) \right\|_K^s |x^v|_K \, |dx| \\ &= b_J^s \, q^{-\sum_{i=1}^{e} y_i (\sigma(a_i) + \langle v, a_i\rangle + m(a_i)s)} |du| , \end{aligned} \tag{3.19}$$

where b_J is a constant, and $y_i \geqslant \alpha_0 + 1$, $i \in J$. Finally, (3.11), (3.19), and (3.17) imply that

$$I(s, E_{A_J}) = \int_{E_{A_J}} \left\| \left(c_1 x^{N_1}, ..., c_l x^{N_l}\right) \right\|_K^s |x^v|_K \, |dx|$$

$$= b_J^s (1 - q^{-1})^n \sum_{\substack{y_i \leq \alpha_0 \\ i \notin J}} \sum_{\substack{y_i \geqslant \alpha_0 + 1 \\ i \in J}} q^{-\sum_{i=1}^{e} y_i (\sigma(a_i) + \langle v, a_i\rangle + m(a_i)s)}$$

$$= b_J^s (1 - q^{-1})^n (Q_J(q^{-s})) \prod_{i\in J} \left(\frac{q^{-(\alpha_0+1)(\sigma(a_i) + \langle v, a_i\rangle + m(a_i)s)}}{1 - q^{-(\sigma(a_i) + \langle v, a_i\rangle + m(a_i)s)}} \right), \tag{3.20}$$

where $Q_J(q^{-s})$ is a polynomial in q^{-s}. ∎

3.1. **Remarks.** (1) For $l=1$ the previous lemma yields to a well-know result about $p-$adic elementary integrals (see e.g. [10, Lemma 8.2.1]). (2) Given a Newton polyhedron $\Gamma(N)\subseteq\mathbb{R}^n$, the data $\sigma(a(\gamma))$, $m(a(\gamma))$, with $a(\gamma)\in\mathfrak{D}(\Gamma(N))$, are invariant under any renaming of the coordinates of $\mathbb{R}^n$. Thus by the considerations made at the end of subsection 2.1, we can take any local coordinate system in the computation of each (N,v). (3) Lemma (3.1) can be extended to integrals over sets of the form $b+(\mathcal{P}_K^{e_0})^n$, with $b\in\mathcal{O}_K^n$. Indeed, if $b\notin(\mathcal{P}_K^{e_0})^n$ and $\left|x^{N_i}\right|_K|_{b+(\mathcal{P}_K^{e_0})^n}$ is constant for some i, the integral has an analytic continuation to the complex plane as a polynomial in q^{-s}. In the other case, by a change of variables, the integral can be reduced to an integral of type (3.1).

4. Local zeta function for algebraic sets

The first result of this paper is the following.

Theorem 4.1. *Let $f_i(x)\in K[x]$, $x=(x_1,...,x_n)$, $f_i(0)=0, i=1,.,l$, be non-constant polynomials, and $f=(f_1,...,f_l)$. Let Φ be a Bruhat-Schwartz function. The local zeta function $Z_\Phi(s,f)$ admits a meromorphic continuation to the complex plane as a rational function of q^{-s} with poles in the set*

$$\bigcup_{(N,v)}\bigcup_{a\in\mathfrak{D}(\Gamma(N))}\left\{-\frac{\sigma(a)+\ \langle v,a\rangle}{m(a)}+\frac{2\pi\sqrt{-1}k}{m(a)\log q},\ k\in\mathbb{Z}\right\}, \tag{4.1}$$

where (N,v) runs over the numerical data of f, and $\langle a,x\rangle=m(a)$, $m(a)\neq0$, is the supporting hyperplane of the facet of $\Gamma(N)$ that corresponds to $a\in\mathfrak{D}(\Gamma(N))$.

Proof. By applying Hironaka's resolution theorem to the divisor $D_K=\cup_{i=1}^l f_i^{-1}(0)$, we get $h:X_K\to A_K^n$ with h a proper $K-$analytic map and X_K a $n-$dimensional $K-$analytic manifold. At every point b of X_K we can choose a chart (U,ϕ_U) such that U contains b, $\phi_U(y)=(y_1,...,y_n)$ and

$$\begin{aligned} f_i\circ h &= \epsilon_i\prod_{j\in J}y_j^{N_{i,j}},\ i=1,...,l, \\ h^*(\bigwedge_{1\le j\le n}dx_j) &= \left(\eta\prod_{j\in J}y_j^{v_j-1}\right)\bigwedge_{1\le j\le n}dy_j, \end{aligned} \tag{4.2}$$

with $J=\{j\in T\mid b\in E_j\}$, and ϵ_i,η units of the local ring $\mathcal{O}_b$ of X_K at b. Since h is proper and $A=Supp(\Phi)$ is compact open, we see that $B=h^{-1}(A)$ is compact open of X_K. Therefore, we can express B as a finite disjoint union of compact open sets B_α in X_K, satisfying $B_\alpha\subseteq U_\alpha$. Since Φ, $|\epsilon_i|_K,|\eta|_K$, are locally constant, after subdividing B_α, we may assume that $(\Phi\circ h)\mid_{B_\alpha}=\Phi(h(b))$, $|\epsilon_i|_K\mid_{B_\alpha}=|\epsilon_i(b)|_K$, $|\eta|_K\mid_{B_\alpha}=|\eta(b)|_K$, and further $\phi_{U_\alpha}(B_\alpha)=D_\alpha$, with $D_\alpha=w+\pi^{e_0}\mathcal{O}_K^n$ for some $w\in K^n$. Since $h:X_K\setminus h^{-1}(D_K)\to A_K^n\setminus D_K$ is $K-$bianalytic, we then have

$$Z_\Phi(s,f)=\sum_\alpha\Phi(h(b))\,|\eta(b)|_K\int_{D_\alpha}\left\|\left(|\epsilon_1(b)|_K\,y^{N_1},...,|\epsilon_l(b)|_K\,y^{N_l}\right)\right\|_K^s|y^v|_K\,|dy|, \tag{4.3}$$

where $N_i:=(N_{i,1},...,N_{i,n})\in\mathbb{N}^n, i=1,...,l$, $N:=(N_1,...,N_l)$, $v:=(v_1-1,...,v_n-1)\in\mathbb{N}^n$, and $N_{i,j}=0$, $v_j=1$, for j not in J. The result follows by applying lemma 3.1, and remark (3.1) (3) to the integral in the right side of (4.3). ∎

The proof of the main theorem is a generalization of Igusa's proof for the case $l=1$, (see [10, Theorem 8.2.1]).

4.1. **Remark.** Since the support of Φ is compact and h is proper the set of numerical data is finite.

5. Twisted local zeta function for algebraic sets

Let χ_i be a character of $\mathcal{O}_K^{\times}$, i.e. a homomorphism $\chi_i : \mathcal{O}_K^{\times} \to \mathbb{C}^{\times}$ with finite image, $i = 1, ..., l$. We formally put $\chi_i(0) = 0$, $i = 1, ..., l$, and $\chi := (\chi_1, ..., \chi_l)$. If $f = (f_1, ..., f_l)$, we define

$$\chi(ac\ f(x)) := \prod_{i=1}^{l} \chi_i(ac\ f_i(x)),$$

where $ac\ z = z\pi^{-v(z)}$ denotes the angular component of $z \in K$. With the above notation, we associate to χ and f the following twisted local zeta function

$$Z_\Phi(s, \chi, f) := \int_{K^n} \Phi(x)\chi(ac\ f(x)) \left\|f(x)\right\|_K^s \mid dx \mid, \qquad s \in \mathbb{C}, \tag{5.1}$$

for $Re(s) > 0$, where Φ is a Schwartz-Bruhat function, and $\mid dx \mid$ denotes the Haar measure on K^nso normalized that $\mathcal{O}_K^n$ has measure 1. The proof of the following result is a simple generalization of the proof of theorem 4.1.

Theorem 5.1. *Let $f_i(x) \in K[x]$, $x = (x_1, ..., x_n)$, $f_i(0) = 0, i = 1, ., l$, be non-constant polynomials, $f = (f_1, ..., f_l)$, and $\chi = (\chi_1, ..., \chi_l)$. Let Φ be a Bruhat-Schwartz function. The local zeta function $Z_\Phi(s, \chi, f)$ admits a meromorphic continuation to the complex plane as a rational function of q^{-s} with poles in the set*

$$\bigcup_{(N,v)} \bigcup_{a \in \mathfrak{D}(\Gamma(N))} \left\{ -\frac{\sigma(a) + \ \langle v, a\rangle}{m(a)} + \frac{2\pi\sqrt{-1}k}{m(a)\log q},\ k \in \mathbb{Z} \right\}, \tag{5.2}$$

where $(N, v) = (\{N_1, ..., N_l\}, v)$, runs over the numerical data of f, and $\langle a, x\rangle = m(a)$, $m(a) \neq 0$, is the supporting hyperplane of the facet of $\Gamma_{(N,v)}$ that corresponds to $a \in \mathfrak{D}(\Gamma(N))$.

6. Local zeta Functions For Monomial Algebraic Sets

We fix $\{N_i \in \mathbb{N}^n \mid N_i \neq 0, i = 1, ..., l\}$, and denote by $\Gamma(\{N_1, ..., N_l\})$ its Newton polyhedron. If $g(x) = \sum_l a_l x^l \in K[x]$, $x = (x_1, .., x_m)$, is a non-constant polynomial satisfying $g(0) = 0$, we set $\mathrm{supp}(g) := \{l \in \mathbb{N}^m \mid a_l \neq 0\}$.

Definition 6.1. *Let $f_i(x) \in K[x]$, $x = (x_1, ..., x_n)$, $f_i(0) = 0, i = 1, ..., l$, be non-constant polynomials. The $K-$algebraic set*

$$V_K(K) = V_K = \{z \in K^n \mid f_i(z) = 0, i = 1, ..., l\}$$

is called a monomial algebraic set if the following conditions hold:

(1) the polynomials $f_i(x)$ have the form

$$f_i(x) = c_i x^{N_i} + g_i(x),\ i = 1, ..., l, \tag{6.1}$$

with $c_i \in \mathcal{O}_K^{\times}$, $N_i \in \mathbb{N}^n$, $N_i \neq 0$, $g_i(x) \in \mathcal{O}_K[x], i = 1, ..., l$;

(2) any $m \in \cup_{i=1}^{l} supp(g_i)$ *belongs to the interior of* $\Gamma(\{N_1, ..., N_l\})$*, in the usual topology of* $\mathbb{R}^n$*;*

(3) the $K-$*singular locus of* V_K *is contained in*

$$\bigcup_{i=1}^{l}\left\{z \in K^n \mid z^{N_i} = 0\right\}.$$

We define $\Gamma_{V_K} := \Gamma(\{N_1, ..., N_l\})$ to be the Newton polyhedron of V_K. Also, we associate to V_K the local zeta function

$$Z(s, V_K) = \int_{\mathcal{O}_K^n} \|f_1(x), ..., f_l(x)\|_K^s \mid dx \mid, \quad \mathrm{Re}(s) > 0. \tag{6.2}$$

The following theorem describes explicitly the meromorphic continuation of $Z(s, V_K)$ in terms of Γ_{V_K}, its proof is a simple modification of the proof of lemma 3.1.

Theorem 6.1. *Let* V_K *a monomial* $K-$*algebraic set with Newton polyhedron* Γ_{V_K}*. Fix a simple polyhedral subdivision of* $\mathbb{R}_+^n$*:*

$$\mathbb{R}_+^n = \{0\} \bigcup \bigcup_{\tau} \left(\bigcup_{i=1}^{l_\tau} \Delta_{\tau,i}\right), \tag{6.3}$$

where τ *is a proper face of* Γ_{V_K}*, and each* $\Delta_{\tau,i}$ *is a simple cone having* τ *as their first meet locus. Then*

$$Z(s, V_K) = (1 - q^{-1})^n + \sum_{\tau} \sum_{i=1}^{l_\tau} \left(\prod_{j=1}^{e_{\tau,i}} \frac{q^{-(\sigma(a_j)+m(a_j)s)}}{1 - q^{-(\sigma(a_j)+m(a_j)s)}}\right), \tag{6.4}$$

where τ *is a proper face of* Γ_{V_K}*, and* $a_1, ..., a_{e_{\tau,i}}$*, are the generators of the cone* $\Delta_{\tau,i}$*.*

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